Lecture 5.

Tuesday, January 28, 2020 6:05 AM

<u>Recall</u>. • <u>Thm1</u>: $\bar\partial$-equation w/ compact support.

• Finish pf of Thm 1.

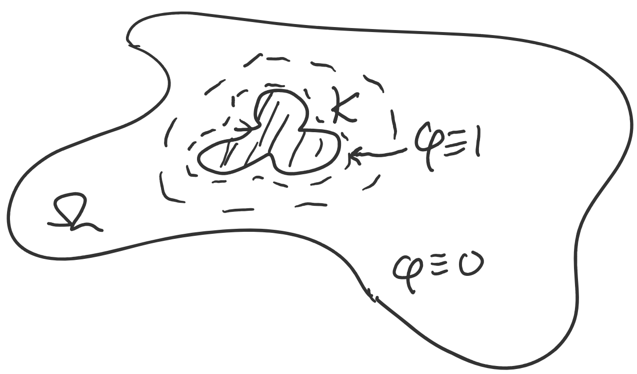
<u>Hartogs Thm-II</u>. Let $\Omega \subseteq \mathbb{C}^n$ domain, $\boxed{n>1}$, $K \subset\subset \Omega$ compact such that $\Omega \setminus K$ is connected. If u is holomorphic in $\Omega\setminus K$, then u extends to a holom. fcn in Ω. ($\exists U$ holom. in Ω s.t. $U|_{\Omega\setminus K} = u$.)

<u>Remark</u>. Spectacularly false in $\mathbb{C}$. $u(z) = \frac{1}{z}$ is holom. in $\mathbb{D}\setminus\{0\}$, $K = \{0\} \subset\subset \mathbb{D}$, but u does not extend to $\mathbb{D}$.

<u>Pf.</u> Let $\varphi \in C_0^\infty(\Omega)$ s.t. $\varphi \equiv 1$ on an open nbhd of K.

Consider $u_0 \in C^\infty(\Omega)$ given by

$$u_0(z) = \begin{cases} (1-\varphi(z))u(z), & z \in \Omega\setminus K \\ 0, & z \in K \end{cases}$$



$\boxed{\text{Thus, } u_0 = u \text{ on } \Omega\setminus \operatorname{supp}\varphi.}$

Look for $U = u_0 + v$ s.t. U holom. in Ω, $U = u_0$ in some open subset of $\Omega\setminus\operatorname{supp}\varphi$. Since $\Omega\setminus\operatorname{supp}\varphi \subseteq \Omega\setminus K$, $\Omega\setminus K$ is connected, this will imply $U = u$ on $\Omega\setminus K$ as desired.

Now, U holom. $\Leftrightarrow \bar\partial U = 0 \Leftrightarrow \bar\partial v = -\bar\partial u_0$. Let $f = -\bar\partial u_0$.

Note that $\bar\partial f = -\bar\partial^2 u_0 = 0$. Moreover, since $\operatorname{supp} f \subseteq \operatorname{supp}\varphi\setminus K \subset\subset \Omega$, Note that $\bar\partial f = -\bar\partial^2 u_0 = 0$. Moreover, since $\operatorname{supp} f \subseteq \operatorname{supp}\varphi \setminus K \subset\subset \Omega$, we may extend f to $\mathcal{C}_0^\infty(\mathbb{C}^n)$ by defining it to be 0 in $\mathbb{C}^n \setminus \Omega$. Clearly this preserves $\bar\partial f = 0$. Now, by Thm 1, we can solve $\bar\partial v = f$ (since $n>1$) in $\mathbb{C}^n$, and $v \in \mathcal{C}_0^\infty(\mathbb{C}^n)$ w/ $v \equiv 0$ in unbdd component G_∞ of $\mathbb{C}^n \setminus \operatorname{supp} f$. With this this choice of v, U is holom. in Ω and $U = u_0$ in $G_\infty \cap \Omega$, which intersects $\Omega \setminus K$. This completes the pf. $\boxtimes$

<u>Prop 1.</u> Let u be holom. in a polydisk $D^n \subseteq \mathbb{C}^n$ and let $Z_u := \{z \in D^n : u(z) = 0\}$. If $Z_u \neq \emptyset$, then Z_u is <u>not</u> compact in D^n!

<u>Rem.</u> Clearly not true in a disk $D \subseteq \mathbb{C}$.

<u>Pf.</u> Suppose $Z_u \neq \emptyset$, $Z_u \subset\subset D^n$ compact. Then $\exists$ smaller polydisk $E^n = E_1 \times \ldots \times E_n$, $\overline{E_j} \subseteq D_j$, such that $Z_u \subseteq E^n$. But $K = \overline{E^n}$ is compact in D^n, $D^n \setminus K$ connected, and $v = \frac{1}{u}$ is holomorphic in $D^n \setminus K$. By Hartogs-II, v is holomorphic in D^n, i.e. across Z_u, which is easily seen to be impossible. $\boxtimes$

<u>Boundary version of Hartogs-II.</u>

<u>Thm 2</u> Let $\Omega \subseteq \mathbb{C}^n$ be bdd, $\mathbb{C}^n \setminus \overline{\Omega}$ connected, and $\partial\Omega$ smooth, i.e. $\exists \rho \in \mathcal{C}^\infty(\mathbb{C}^n; \mathbb{R})$ s.t. $\partial\Omega = \{z \in \mathbb{C}^n : \rho(z) = 0\}$, $d\rho \neq 0$ on $\partial\Omega$ (a <u>defining fcn</u> for $\partial\Omega$). If $u \in \mathcal{C}^\infty(\overline{\Omega})$ and $\bar\partial u \wedge \bar\partial\rho = 0$ on $\partial\Omega$ ($u|_{\partial\Omega}$ is CR), then $\exists\, U \in \mathcal{C}^\infty(\overline{\Omega})$

$\bar\partial u \wedge \bar\partial \rho = 0$ on $\partial\Omega$ ($u|_{\partial\Omega}$ is CR), then $\exists\, U \in \mathcal{C}^\infty(\bar\Omega)$ s.t. U holom. in Ω and $U|_{\partial\Omega} = u$.

Rem. ① If such U exists, $U - u = a\rho \Rightarrow \bar\partial U - \bar\partial u = a\bar\partial\rho$ on $\partial\Omega = \{\rho = 0\}$. Thus, since U holom., $\bar\partial u \wedge \bar\partial\rho = a\bar\partial\rho\wedge\bar\partial\rho = 0$.

② Ex. $\bar\partial u \wedge \bar\partial u$ can be reformulated as

(CR) $\forall z \in \partial\Omega$: $\sum_{j=1}^{n} c_j \frac{\partial u}{\partial \bar z_j}(z) = 0$, $\forall c = (c_1, \dots, c_n)$: $\sum_{j=1}^{n} \frac{\partial \rho}{\partial \bar z_j}(z)\, c_j = 0$

This is a condition that depends only on $u|_{\partial\Omega}$! Functions $u \in \mathcal{C}^\infty(\partial\Omega)$ that satisfy (CR) are called CR functions.